Normal and anti-normal ordered expressions for annihilation and creation operators

J.M. Vargas-Martínez and H. Moya-Cessa
Instituto Nacional de Astrofísica, Optica y Electrónica,
Apdo. Postal 51 y 216, 72000 Puebla, Pue., Mexico
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We give the normal and anti-normal order expressions of the number operator to the power k by using the commutation relation between the annihilation and creation operators. We use those expressions to give general formulae for functions of the number operator in normal and anti-normal order.

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I. INTRODUCTION

In some problems in quantum mechanics it is needed to calculate functions of the operator $\hat{n} = \hat{a}^\dagger \hat{a}$ where $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators of the harmonic oscillator, respectively. For instance in ion traps [1] it is usual to have associated Laguerre polynomials of order $\hat{n}$ [2, 3].

Very recently Fujii and Suzuki have shown ordering expressions for $\hat{n}^k$ as different types of polynomials with respect to the number operator [4]. They have shown nontrivial relations including the use of Stirling numbers of the first kind [5].

Here we in fact do the opposite: we obtain an expression for $\hat{n}^k$ in normal order (the antinormal order is then straightforward, as it will be given in terms of similar coefficients [6]), i.e. a sum of coefficients multiplying normal ordered forms of $\hat{a}$ and $\hat{a}^\dagger$. This allows us to obtain an expression for the normal ordered form of a function of the operator $\hat{n}$ and demonstrate as a particular example a lemma in Louissel's book for the exponential of the number operator [7].

II. NORMAL ORDERING

One may use the commutation relations of the annihilation and creation operators to obtain the powers of $\hat{n}$ in normal, antinormal or symmetric order [7]. For instance, we can express $\hat{n}^k$ in normal order, for $k = 2$ as

$$\hat{n}^2 = [\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}, \tag{1}$$

for $k = 3$ as

$$\hat{n}^3 = [\hat{a}^\dagger]^3 \hat{a}^3 + 3[\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}, \tag{2}$$

and for $k = 4$

$$\hat{n}^4 = [\hat{a}^\dagger]^4 \hat{a}^4 + 6[\hat{a}^\dagger]^3 \hat{a}^3 + 7[\hat{a}^\dagger]^2 \hat{a}^2 + \hat{a}^\dagger \hat{a}, \tag{3}$$

where the coefficients multiplying the different powers of the normal ordered operators do not show an obvious form to be determined. In writing the above equations we have used repeatedly the commutator $[\hat{a}, \hat{a}^\dagger] = 1$. We may infer that the coefficients in the above equations are Stirling numbers of the second kind (see also [8]), i.e. we obtain

$$\hat{n}^k = \sum_{m=0}^{k} S_k^{(m)} [\hat{a}^\dagger]^m \hat{a}^m, \tag{4}$$

with [5]

$$S_k^{(m)} = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \frac{m!}{j!(m-j)!} j^k. \tag{5}$$

We now write a function of $\hat{n}$ in a Taylor series as

$$f(\hat{n}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \hat{n}^k, \tag{6}$$

and inserting (4) in this equation we obtain

$$f(\hat{n}) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{m=0}^{k} S_k^{(m)} [\hat{a}^\dagger]^m \hat{a}^m. \tag{7}$$

Because $S_k^{(m)} = 0$ for $m > k$ we can take the second sum in (7) to infinite and interchange the sums to have

$$f(\hat{n}) = \sum_{m=0}^{\infty} [\hat{a}^\dagger]^m \hat{a}^m \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} S_k^{(m)}. \tag{8}$$

For the same reason stated above, we may start the second sum at $k = m$,

$$f(\hat{n}) = \sum_{m=0}^{\infty} [\hat{a}^\dagger]^m \hat{a}^m \sum_{k=m}^{\infty} \frac{f^{(k)}(0)}{k!} S_k^{(m)}. \tag{9}$$

By noting that

$$\frac{\Delta^m f(x)}{m!} = \sum_{k=m}^{\infty} \frac{f^{(k)}(x)}{k!} S_k^{(m)}, \tag{10}$$

where Δ is the difference operator, defined as [5]

$$\Delta^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} f(x+k), \tag{11}$$

we may write (9) as

$$f(\hat{n}) = \sum_{m=0}^{\infty} \frac{[\hat{a}^\dagger]^m \hat{a}^m \Delta^m}{m!} f(0) \equiv : e^{\Delta \hat{n}} : f(0) \tag{12}$$

where $: \hat{n} :$ stands for normal order.

A. Lemma 1

If we choose the function $f(\hat{n}) = \exp(-\gamma \hat{n})$, we have that

$$\Delta^m f(0) = \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!(m-k)!} e^{-\gamma k}, \tag{13}$$

and then we obtain the well-known lemma [7]

$$e^{-\gamma \hat{n}} = : e^{(e^{-\gamma} - 1)\hat{n}} : . \tag{14}$$

III. ANTI-NORMAL ORDERING

Following the procedure introduced in the former section, we can write $\hat{n}^k$ in anti-normal order as

$$\hat{n}^k = (-1)^k \sum_{m=0}^{k} (-1)^m S_{k+1}^{(m+1)} \hat{a}^m [\hat{a}^\dagger]^m, \tag{15}$$

and a function of the number operator as

$$f(\hat{n}) = \sum_{m=0}^{\infty} (-1)^m \hat{a}^m [\hat{a}^\dagger]^m \sum_{k=m}^{\infty} (-1)^k \frac{f^{(k)}(0)}{k!} S_{k+1}^{(m+1)}. \tag{16}$$

The second sum differs from (10) in the extra $(-1)^k$ and the parameters of the Stirling numbers. We can define $u = -x$, such that $f^{(k)}(x)_{x=0} = (-1)^k f^{(k)}(u)_{u=0}$, and use the identity [5]

$$S_{k+1}^{(m+1)} = (m+1)S_k^{(m+1)} + S_k^{(m)} \tag{17}$$

to write

$$\begin{aligned} f(\hat{n}) &= \sum_{m=0}^{\infty}(-1)^m \hat{a}^m [\hat{a}^\dagger]^m \\ &\left((m+1)\sum_{k=m}^{\infty}\frac{f^{(k)}(u=0)}{k!}S_k^{(m+1)} + \sum_{k=m}^{\infty}\frac{f^{(k)}(u=0)}{k!}S_k^{(m)}\right) \end{aligned} \tag{18}$$

so we can use again Eq. (10) to finally write

$$f(\hat{n}) = (1+\Delta)\vdots e^{-\Delta\hat{n}}\vdots f(0) \tag{19}$$

where $\vdots\hat{n}\vdots$ stands for anti-normal order.

A. Lemma 2

Let us consider again the function $f(\hat{n}) = \exp(-\gamma\hat{n})$. This gives us that $f(x) = e^{-\gamma x}$ and $f(u) = e^{\gamma u}$. Therefore

$$\Delta^m f(u=0) = \sum_{k=0}^{m}(-1)^{m-k}\frac{m!}{k!(m-k)!}e^{\gamma k} = (e^\gamma - 1)^m, \tag{20}$$

such that we can obtain the exponential of the number operator in anti-normal order (lemma) as

$$e^{-\gamma\hat{n}} = e^{\gamma}\vdots e^{(1-e^\gamma)\hat{n}}\vdots. \tag{21}$$

1. *Coherent states.*

Let us use Eq. (21) to find averages for coherent states, $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$, where $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ is the so-called displacement operator and $|0\rangle$ is the vacuum state:

$$\langle\alpha|e^{-\gamma\hat{n}}|\alpha\rangle = e^{\gamma}\langle\alpha|\sum_{m=0}^{\infty}\frac{(1-e^\gamma)^m}{m!}\hat{a}^m[\hat{a}^\dagger]^m|\alpha\rangle \tag{22}$$

by using that

$$\begin{aligned} \langle\alpha|\hat{a}^m[\hat{a}^\dagger]^m|\alpha\rangle &= \langle 0|(\hat{a}+\alpha)^m(\hat{a}^\dagger+\alpha^*)^m|0\rangle \\ &= \sum_{k=0}^{m}|\alpha|^{2k}\left(\frac{m!}{(m-k)!k!}\right)^2(m-k)! \end{aligned} \tag{23}$$

we may write

$$\langle\alpha|e^{-\gamma\hat{n}}|\alpha\rangle = e^{\gamma}\sum_{m=0}^{\infty}(1-e^\gamma)^m L_m(-|\alpha|^2) \tag{24}$$

where $L_m(x)$ are the Laguerre polynomials of order m. We can finally write a closed expression for the sum above [9] to obtain the expected result for coherent states

$$\langle\alpha|e^{-\gamma\hat{n}}|\alpha\rangle = e^{|\alpha|^2(e^{-\gamma}-1)}. \tag{25}$$

2. Fock states.

For Fock or number states we obtain

$$\begin{aligned} \langle n|e^{-\gamma\hat{n}}|n\rangle &= e^{\gamma}\langle n|\sum_{m=0}^{\infty}\frac{(1-e^{\gamma})^m}{m!}\hat{a}^m[\hat{a}^{\dagger}]^m|n\rangle \\ &= e^{\gamma}\sum_{m=0}^{\infty}\frac{(1-e^{\gamma})^m}{m!}\frac{(m+n)!}{n!} \end{aligned} \tag{26}$$

rearranging the sum above with $k = n + m$ we have

$$\langle n|e^{-\gamma\hat{n}}|n\rangle = e^{\gamma}\sum_{k=n}^{\infty}(1-e^{\gamma})^{k-n}\frac{k!}{n!(k-n)!} \tag{27}$$

which has a closed expression, as $\sum_{k=n}^{\infty} x^{k-n}\frac{k!}{n!(k-n)!} = (1-x)^{-n-1}$ [5]:

$$\langle n|e^{-\gamma\hat{n}}|n\rangle = e^{-\gamma n} \tag{28}$$

IV. CONCLUSIONS

In conclusion, we have written the normal and anti-normal order expressions of $\hat{n}^k$ by using the commutation relation between the annihilation and creation operators. The coefficients for such expressions are the Stirling numbers of the second kind [8]. We then have used the difference operator to write a function (that may be developed in Taylor series) of the number operator in normal and anti-normal order, showing consistency with the particular case of the exponential function lemma in normal order.

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